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NOTE ON THE EXPANSION OF A FUNCTION.

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1. The following simple method of obtaining the expansion of a function in integral powers of the variable I have not seen in print.

Differentiate the function

$$\frac{fx - fa}{x - a} \tag{1}$$

n times with respect to a , by applying Leibnitz's formula for forming the n th derivative of a product, and we obtain at once

$$\begin{aligned} fx - fa - (x - a)f'a - \dots - \frac{(x - a)^n}{n!}f^n a \\ = \frac{(x - a)^{n+1}}{n!} \left[\frac{d}{da} \right]^n \left[\frac{fx - fa}{x - a} \right] \end{aligned} \tag{2}$$

in which identity, the member on the right represents the remainder.

Since the above is a mere algebraical identity, it is true, whether the argument be real or complex. The only consideration we have to make in regard to the function f , is that its n derivatives are finite and existent at a .

2. To throw the remainder into familiar form we make use of the theorem of mean value, which may be presented thus :—

We have,

$$\frac{fx - fa}{x - a} = \frac{1}{m} \left[\frac{fx - fx_1}{\Delta x} + \frac{fx_1 - fx_2}{\Delta x} + \dots + \frac{fx_{n-1} - fa}{\Delta x} \right], \tag{3}$$

in which we have $x - a = m\Delta x$, and have assumed that along the line between x and a the function f is finite at the points $x_r = x + r\Delta x$ ($r = 0, \dots, n$). Evidently the second member of (3) is the mean of the m ratios in the parenthesis; if the arguments are real, the left member lies between the greatest and least of these ratios. If the derivative of the function f is not infinite throughout the interval between x and a we may make m as great as we choose, and in the limit, when $n = \infty$, we find that the ratio $(fx - fa)/(x - a)$ must lie between the greatest and least values of the derivative of f , in the interval (x, a) . If the derivative be continuous between x and a , or only continuous between those two values of the argument in the interval (x, a) at which it takes the greatest and least values, then evidently there must be a value of the

argument u , between these values, such that

$$\frac{fx - fa}{x - a} = f'u. \quad (4)$$

If the argument of the function be complex, then we have from (3)

$$\left| \frac{fx - fa}{x - a} \right| = \frac{1}{m} \left| \sum \frac{dfx}{dx} \right|$$

$$< \frac{1}{m} \sum \left| \frac{dfx}{dx} \right| = \frac{\rho}{m} \sum \left| \frac{dfx}{dx} \right| = \rho |f'u|.$$

In which ρ is a real positive number less than unity, and u is some point on the straight line between x and a . Let φ be the amplitude of the member on the left and ψ that of $f'u$, then

$$\frac{fx - fa}{x - a} = \rho e^{i(\phi - \psi)} f'u = \lambda f'u \quad (5)$$

wherein $\text{mod } \lambda$ is less than unity. This is then the theorem of mean value, and when the argument is real $\lambda \equiv 1$.

The remainder (2) may now be written

$$\frac{(x - a)^{n+1}}{n!} \left[\frac{d}{da} \right]^n \left[\frac{fx - fa}{x - a} \right] = \frac{(x - a)^{n+1}}{n!} \left[\frac{d}{da} \right]^n (\lambda f'u)$$

$$= \frac{(x - a)^{n+1}}{n!} \left[\frac{d}{da} \right]^n \lambda f'[x + \theta(a - x)], \quad (6)$$

θ being a real positive number less than unity.

3. We now show that the differentiation indicated in (6) may be performed just as though λ and θ were absolute constants with respect to a . As in Todhunter's Calculus and elsewhere, let the right member of (2) be represented by $R(a)$. We have $R(x) = 0$, and by differentiating with respect to a , we get

$$R'(a) = -\frac{(x - a)^n}{n!} f^{n+1}a.$$

Applying the theorem of mean value to the function $R(a)$, we have

$$R(a) = \lambda' (a - x) R'(u)$$

$$= \frac{(x - a)(u' - a)^n}{n!} \lambda' f^{n+1}u$$

$$= \frac{(x - a)^{n+1}}{n!} \lambda' \theta^n f^{n+1} [x + \theta'(a - x)],$$

in which, as before, all we know of λ' and θ' is that $|\lambda'| < 1$ and $0 < \theta' < 1$, and if x and a are real $\lambda' \equiv 1$.

We conclude, therefore, that in order to derive the Taylor-Cauchy formula for either real or complex variables, it is only necessary to differentiate the theorem of mean value

$$\frac{f x - f a}{x - a} = \lambda f' [x - \theta (a - x)]$$

n times with respect to a , regarding λ and θ as constants, which however change their values during differentiation, but always remain such that $|\lambda| < 1$, $0 < \theta < 1$, and $\lambda \equiv 1$ when x and a are real.

